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THE CAYLEYAN CURVE OF THE QUARTIC

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The quartic curve $(\alpha x)^4 = 0$ determines a correspondence

$$(\alpha x)^2(\alpha y)\alpha_i = 0, \qquad i = 0, 1, 2,$$

in which x is a point on the Hessian curve and y a point of the Steinerian. The locus of lines ξ joining corresponding points x, y is the Cayleyan, known to be of degree 18 in ξ and 12 in the coefficients of the quartic.

The Cayleyan can be expressed in terms of the two contravariants of the quartic, $(s\xi)^4$ $[\equiv \frac{1}{2}|\alpha\beta\xi|^4]$ and $(t\xi)^6$ $[\equiv \frac{1}{6}|\beta\gamma\xi|^2|\gamma\alpha\xi|^2|\alpha\beta\xi|^2]$, and of terms produced by operating with the polars of these on $(\alpha x)^4$. The working out of this depends on a special reference triangle which is always valid for the general quartic. Suppose

$$\begin{aligned}(ax)^4 \equiv\ & ax_0^4 + 4a_1x_0^3x_1 + 4a_2x_0^3x_2 + 6hx_0^2x_1^2 + 12lx_0^2x_1x_2 + 6gx_0^2x_2^2 \\ & + 4b_0x_0x_1^3 + 12mx_0x_1^2x_2 + 12nx_0x_1x_2^2 + 4c_0x_0x_2^3 \\ & + bx_1^4 + 4b_2x_1^3x_2 + 6fx_1^2x_2^2 + 4c_1x_1x_2^3 + cx_2^4.\end{aligned}$$

Since x and y as given above can never coincide for the general quartic, because

$$(\alpha x)^3\alpha_i = 0, \qquad i = 0, 1, 2,$$

is the condition that $(\alpha x)^4$ have a double point, let x, the point of the Hessian, be the reference point $(0, 1, 0)$ and let y, the point of the Steinerian, be $(0, 0, 1)$, so that $x_0 = 0$ is a line of the Cayleyan. Then

$$\begin{aligned}\alpha_1^2\alpha_2\alpha_i &= 0, \qquad i = 0, 1, 2, \\ \text{or} \quad m &= b_2 = f = 0.\end{aligned}$$

This reference scheme is maintained throughout, though more highly specialized as occasion demands. Under it the Hessian becomes

$2n(bh - b_0^2)x_1^5x_0 + 2c_1(bh - b_0^2)x_1^5x_2 +$ lower powers of x_1;

$(s\xi)^4 \equiv bc\xi_0^4 - 4cb_0\xi_0^3\xi_1 + 4(-bc_0 + b_0c_1)\xi_0^3\xi_2 + \ldots\ldots;$

$(t\xi)^6 \equiv -bc_1^3\xi_0^6 + 2(-bcn + bc_0c_1 + 2b_0c_1^2)\xi_0^5\xi_1 + 6bc_1n\xi_0^5\xi_2 + \ldots\ldots$

In the first place, the Cayleyan is known to be on the stationary lines of the quartic, which are the common lines of $(s\xi)^4$ and $(t\xi)^6$. Therefore the Cayleyan must be made up of terms containing either $(s\xi)^4$ or $(t\xi)^6$ at least once.

Now let us see what are the common lines of the Cayleyan and $(s\xi)^4$. To make x_0 a line of the latter requires that $bc = 0$. If $b = 0$, then x_0 is a stationary line of the quartic. If $c = 0$, then not only is x_0 a line of $(s\xi)^4$, but its contact with it is $(0, 0, 1)$, the point of the Steinerian, which has also become a point of the quartic. Therefore quartic, Steinerian, and $(s\xi)^4$ all meet in a point. The Steinerian, a curve of order 12, meets the quartic in 48 points; the 48 corresponding lines together with the 24 stationary lines make up the 72 common lines of the Cayleyan and $(s\xi)^4$. The condition that the polar point of $(s\xi)^4$ lie on $(\alpha x)^4$ is the vanishing of $(s\xi)^3(s\alpha)\,(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$; this, when multiplied by $(t\xi)^6$, is of proper degree for a term of the Cayleyan. It is, then, the only term of the Cayleyan not containing $(s\xi)^4$.

It is now in order to ask for the common lines of the Cayleyan and $(t\xi)^6$. For x_0 to be a line of the latter requires that $bc_1^2 = 0$. Again setting aside the stationary lines, we have $c_1 = 0$. Then it is seen that x_0 has as its contact with $(t\xi)^6$ the point $(0, 1, 0)$; furthermore, it is tangent to the Hessian at the same point. Therefore there are a certain number of lines of the Cayleyan which are also lines of both the Hessian and $(t\xi)^6$, these two curves having contact on these lines. For the terms of the Cayleyan not containing $(t\xi)^6$ it is sufficient to use

$$\begin{aligned}
&\lambda(s\xi)^4.(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)\\
&+ (s\xi)^4(s'\xi)^4\,[\sigma(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^4(t\alpha)^2\\
&\quad + \tau(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)^2(t\xi)^5(t\alpha)]\\
&+ (s\xi)^4(s'\xi)^4(s''\xi)^4\,[\varphi(s'''\xi)^3(s'''\alpha)^2(t\xi)^4(t\alpha)^2 + \mu(s'''\xi)^3(s'''\alpha)(t\xi)^3(t\alpha)^3]\\
&+\nu\,(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^4.(t\xi)^2(t\alpha)^4,
\end{aligned}$$

where λ, σ, τ, φ, μ, ν are undetermined coefficients. By requiring the highest power of ξ_0 to vanish when $c_1 = 0$ certain relations on these coefficients are obtained, not enough to solve, however. To the terms given above it is necessary to add terms containing $(t\xi)^6$;

$$\begin{aligned}
&\rho(t\xi)^6\,.\,(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\
&+ \epsilon(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^6\,.\,(s'''\alpha)^4
\end{aligned}$$

will be found sufficient.

There are certain other lines known to be lines of the Cayleyan. There are 21 points whose polar cubics as to the quartic break up into a conic and a line, which is a fourfold line of the Cayleyan. Let one of the 21 points be $(1, 0, 0)$; then for the polar cubic to contain x_0 as a factor requires that $b_0 = c_0 = n = 0$. Now, using this condition, require that the highest power of ξ_0 in the expression for the Cayleyan vanish. The result will be certain conditions on the undetermined coefficients, but still not enough to solve.

Instead of putting the Cayleyan again on these lines it is easier to proceed at once to the general reference scheme which has been the basis of all the work, when only $m = b_2 = f = 0$, and require that the highest coefficient of ξ_0 vanish. This at once completes the work and furnishes proof of its correctness. The Cayleyan is obtained as

$$
\begin{aligned}
&33\,(s\xi)^4\,.\,(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)\\
&-\tfrac{22}{3}\,(t\xi)^6\,.\,(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\
&+(s\xi)^4(s'\xi)^4\,[15(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^4(t\xi)^2\\
&\quad-57(s''\xi)^3(s''\alpha)(s'''\xi)^2(s'''\alpha)^2(t\xi)^5(t\alpha)\,]\\
&+(s\xi)^4(s'\xi)^4(s''\xi)^4[\tfrac{35}{2}\,(s'''\xi)^2[s'''\alpha)^2(t\xi)^4(t\alpha)^2-10(s'''\xi)^3(s'''\alpha)(t\xi)^4(t\alpha)^3]\\
&+\tfrac{65}{18}\,(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^6\,.\,(s\alpha)^4\\
&-\tfrac{5}{3}\,(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^4\,.\,(t\xi)^2(t\alpha)^4.
\end{aligned}
$$

Since, however, this expression has been obtained by causing a coefficient to vanish, there is the possibility that it gives merely a syzygy and vanishes identically. Therefore it was tested on the special quartic $x_0^4 + x_1^4 + x_2^4$, where the Cayleyan is known to be $\xi_0^2\xi_1^2\xi_2^2$, and found not to vanish.

The stationary lines of the quartic are known to be lines of the Steinerian. From the above form of the Cayleyan it can be shown that the contacts of these lines are the same for the Cayleyan as for the Steinerian, so that the two curves touch. We had also certain common lines of the Cayleyan and Hessian, which were likewise lines of $(t\xi)^6$. These lines can be shown to have the same contact as to the three curves. The Cayleyan and $(t\xi)^6$ have 108 common lines, 24 of which are absorbed by the flexes, leaving 84 to be accounted for here. Because of the contact of the curves each line counts for two common lines; therefore the Cayleyan, Hessian, and $(t\xi)^6$ touch in 42 points.

Certain interesting facts come up under the reference scheme here employed. The polar conic of $(0, 1, 0)$ is

$$hx_0^2 + 2b_0x_0x_1 + bx_1^2 = 0,$$

two lines, which coincide if

$$bh - b_0^2 = 0.$$

But this requires the Hessian to have a double point; therefore the general quartic cannot have a polar conic made up of two coincident lines. Also the polar cubic of (0, 0, 1) is

$$a_2x_0^3 + 3lx_0^2x_1 + 3gx_0^2x_2 + 6nx_0x_1x_2 + 3c_0x_0x_2^2 + 3c_1x_2^2 + cx_2^3 = 0.$$

This can have a cusp only if

$$c_1l - n^2 = 0.$$

This has clearly nothing to do with $b = 0$, the condition that x_0 be a stationary line of the quartic. Therefore the cusps of the Steinerian do not lie on the stationary lines, as might be expected from their number —twenty-four. $n = 0$ is the condition that $x_2 = 0$ be the tangent to the Hessian; then the cusp cannot be obtained by making $c_1 = 0$, for then the Hessian has a double point. Putting $l = 0$ shows that the cusp tangent is also the tangent to the Hessian. Use of $n = 0$ also shows that the polar points of lines of the Cayleyan as to $(t\xi)^6$ lie on the corresponding tangents to the Hessian.

A SEARCH FOR AN EINSTEIN RELATIVITY-GRAVITATIONAL EFFECT IN THE SUN

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From the equivalence principle of generalized relativity Einstein[1] concludes that the propagation of light is influenced by gravitation, and deduces two important consequences that can be subjected to the test of observation; namely, a train of light waves passing close to the edge of the sun is refracted so that the angular distance of a star appearing near the sun is increased by 1".75, and the Fraunhofer lines are displaced to the red in the solar spectrum by an amount equivalent to a velocity of recession of 0.634 km/sec. The amount depends only on the difference in gravitational potential between the gravitation field in which the radiation originates and the field where it is received. In the case of massive stars with density comparable to that of the sun the line displacement may be large, equivalent to 0.634 km/sec. $M^{2/3}\,d^{1/3}$, where M and d are in terms of the sun's mass and density.[2] Confirmation of either of these consequences would have not only an important bearing upon the establishment of the relativity principle but also upon the in-

terpretation of astrophysical data. The problem of determining stellar motions in the line of sight, a matter of fundamental importance, would be confronted with difficulties of a high order, depending as it does upon line displacement in stellar, relative to terrestrial spectra. Our knowledge of the motions, pressure, and many other phenomena in the solar atmosphere must be obtained from line displacements in the spectrum, but here it would be possible to apply definite corrections, this would in many cases, however, modify our interpretations.

The question of confirmation may be approached through direct observation of the displacement of a star near the sun[3] but the conditions of observation are beset with great difficulties, statistical study of stellar masses and motions[4] or the determination of displacements of solar lines[5] under conditions that eliminate other possible causes.

Some results of an investigation bearing upon the relativity displacement of lines in the sun's spectrum are here communicated, a full account will appear later in a *Contribution from the Mount Wilson Solar Observatory*.

The substance of the investigation is the behavior of lines in the nitrogen (cyanogen) bands, λ 3883, at the center and at the limb of the sun, these lines owing to their freedom from pressure shift are in so far well fitted for such an investigation, but their compound character in the most important series and the frequent superposition of the lines of different series may introduce disturbing factors. The line density in the bands is high, above ten lines per angstrom. The possible occurrence of blends with metallic lines and the closeness of adjacent lines are important considerations in the selection of lines for observation. As the probability of blends is least for the narrowest lines and the precision of measurement is highest for lines showing no evidence of duplicity and sufficiently separated from adjacent lines,[6] a greater weight is attributed to the lines of group A than to the broader lines of group B in the accompanying table.

The wave-lengths of the lines were measured in the arc and at the center and limb in terms of identical iron standards. The limb-center shifts were found by two independent determinations. The sun-arc displacements at the center were obtained by direct comparison and by three indirect methods. The 'Standard' (R−I) is the mean difference, Rowland minus International for lines whose wave, lengths in the sun and arc are equal, determined from the iron lines in this region by taking account of their sun-arc displacements.

The mean, zero, given by 43 lines for center-arc requires from the relativity point of view a radial movement in the solar atmosphere just

COMPARATIVE SOLAR AND TERRESTRIAL WAVE-LENGTHS OF LINES OF THE NITROGEN (CYANOGEN) BANDS

	GROUP A 25 LINES INTENSITY	GROUP B 18 LINES INTENSITY
At center	00–1	2–4
a. Direct comparison of sun and arc	0.000	+0.0013 A
b. λ at center minus λ in arc	0.000	+0.0026
c. (R–I) for band lines minus "Standard" (R–I)	−0.003	+0.0010
d. (λ limb–λ arc) minus (λ limb–λ center)	−0.001	+0.0008
Mean (Center–Arc)	−0.001	+0.0014
	17 LINES INTENSITY	18 LINES INTENSITY
At limb	00–1	2–4
a. λ at limb minus λ in arc	0.000	+0.0037
b. (λ center–λ arc)+(λ limb–λ center)	0.000	+0.0035
Mean (Limb–Arc)	0.000	+0.0036

balancing the gravitational effect; at the limb where its influence would disappear, a displacement to the red of 0.008 A should be observed. For the lines of highest weight the displacement at the limb is zero, the mean for all lines +0.0018 A. Owing to the probable occurrence of blends in this region, the displacement to the red for the broader lines of group B may be in part at least attributed to this influence, as the effect of blends with metallic lines is to introduce systematic displacement to the red even for lines normally undisplaced at the limb.

The general conclusion from the investigation is that within the limits of error the measurements show no evidence of an effect of the order deduced from the equivalence relativity principle. Two other series of observations at Mount Wilson are at variance with the relativity principle, the equal wave-lengths of the H and K lines of calcium in the arc and at the sun's limb[7] and a center-arc displacement of 0.004 A at λ 6300 for iron lines instead of 0.013 A required by the relativity principle.

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[7] St. John, C. E., *Mt. Wilson Contrib.*, No. 48; *Astroph. J., Chicago*, **32**, 1910, (36–82).

TRIADS OF TRANSFORMATIONS OF CONJUGATE SYSTEMS OF CURVES

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When the rectangular point coordinates x,y,z, of a surface satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{1}$$

the curves $u = \text{const.}$ $v = \text{const.}$ form a conjugate system. We assume that the parametric system is of this sort throughout this note, and we shall speak of the *net* of parametric curves. Equation (1) is the *point equation* of the net.

If N is such a net, a second net N' of coordinates x', y', z', is given by the quadratures

$$\frac{\partial x'}{\partial u} = h'\frac{\partial x}{\partial u}, \qquad \frac{\partial x'}{\partial v} = l'\frac{\partial x}{\partial v}, \tag{2}$$

provided that h' and l' are functions of u and v subject to the conditions

$$\frac{\partial h}{\partial v} = a(l' - h'), \qquad \frac{\partial l'}{\partial u} = b(h' - l'). \tag{3}$$

Moreover, each pair of solutions of these equations leads by (2) to a net N', which is such that the tangents at corresponding points M and M' to the curves of the nets are parallel. All nets *parallel* to N are obtained in this way.

If θ_1 is any solution of (1), and θ_1' is the function given by

$$\frac{\partial\theta_1'}{\partial u} = h'\frac{\partial\theta_1}{\partial u}, \qquad \frac{\partial\theta_1'}{\partial v} = l'\frac{\partial\theta_1}{\partial v}, \tag{4}$$

then the functions $x_1^{(1)}$, $y_1^{(1)}$, $z_1^{(1)}$, defined by equations of the form

$$x_1^{(1)} = x - \frac{\theta_1}{\theta_1'}x' \tag{5}$$

are the coordinates of a net $N_1^{(1)}$, so related to N that the lines joining corresponding points M and $M_1^{(1)}$ of these nets form a congruence whose developables meet the surface on which these nets lie in the curves of the nets. We say that two nets so related geometrically are in the relation of a *transformation* T. Parallel nets are in such relation. We

have shown[1] that any transformation T of N into a non-parallel net is given by equations of the form (5).

Let N'' be a second net parallel to N, its coordinates being given by

$$\frac{\partial x''}{\partial u} = h'' \frac{\partial x}{\partial u}, \qquad \frac{\partial x''}{\partial v} = l'' \frac{\partial x}{\partial v}, \tag{6}$$

and let θ_1'' be defined by

$$\frac{\partial \theta_1''}{\partial u} = h'' \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1''}{\partial v} = l'' \frac{\partial \theta_1}{\partial v}, \tag{7}$$

h'' and l'' being a pair of solutions of (3).

Then a second transform $N_1^{(2)}$ has coordinates of the form

$$x_1^{(2)} = x - \frac{\theta_1}{\theta_1''} x''. \tag{8}$$

Since the nets N' and N'' are parallel to one another, and the functions θ_1' and θ_1'' are solutions of the respective point equations for N' and N'' in a relation analogous to (4), a transformation T of N'' is given by

$$x_1^{(1)'''} = x'' - \frac{\theta_1''}{\theta_1'} x'. \tag{9}$$

We denote by $N_1^{(1)'''}$ the net with these coordinates. By differentiating the expressions (9), we show that the nets $N_1^{(1)}$ and $N_1^{(1)'''}$ are parallel. Moreover, it can be shown that the equations

$$x_1^{(2)} = x_1^{(1)} - \frac{\theta_1}{\theta_1''} x_1^{(1)'''} \tag{10}$$

are consistent with the above equations, and consequently $N_1^{(2)}$ is a T transform of $N_1^{(1)}$. Hence *if a net is transformed into two nets by means of the same function θ, the new nets are in the relation of a transformation T.* We say that three such nets form a *triad* under transformations T. It can be shown that the relation is entirely reciprocal in the sense that any two are obtainable from the third by transformations involving the same solution of the point equation of the third net. If in particular we take for θ_1 any of the coordinates of N, say z, the nets $N_1^{(1)}$ and $N_1^{(2)}$ lie in the plane $z = 0$. In other words, the developables of the two congruences, obtained by drawing through points of a net N lines parallel to the corresponding radii vectors of two nets parallel to N meet any plane in two nets in the relation of a transformation T. (We postpone to a later time a discussion of transformations of planar nets.)

If θ_1 is any solution of (1), a solution of the point equation of $N_1^{(1)}$ is given by

$$\theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1'}\theta_2'. \tag{11}$$

This function and the net $N_1^{(1)'''}$ parallel to $N_1^{(1)}$ determine a transformation of the latter; moreover, the congruence of the transformation consists of the joins of corresponding points on $N_1^{(1)}$ and $N_1^{(2)}$. We call the transform N_{12} and its point coordinates x_{12}, y_{12}, z_{12}. The solution of the point equation of $N_1^{(1)'''}$ corresponding to θ_{12} is of the form

$$\theta_{12}''' = \theta_2''' - \frac{\theta_1'''}{\theta_1'}\theta_2', \tag{12}$$

and consequently we have

$$x_{12} = x_1^{(1)'''} - \frac{\theta_2\theta_1' - \theta_1\theta_2'}{\theta_2'''\theta_1' - \theta_1'''\theta_2'} x_1^{(1)'''}. \tag{13}$$

The function θ_2 can be used to determine with the nets N' and N'' two transforms of N, namely $N_2^{(1)}$ and $N_2^{(2)}$, whose coordinates are of the respective forms

$$x_2^{(1)} = x - \frac{\theta_2}{\theta_2'}x', \qquad x_2^{(2)} = x - \frac{\theta_2}{\theta_2''}x''. \tag{14}$$

Corresponding points of the nets N, $N_1^{(2)}$, and $N_2^{(2)}$ lie on a line, and $N_2^{(2)}$ is a transform of $N_1^{(2)}$ by means of the function

$$\theta_2 - \frac{\theta_1}{\theta_1''}\theta_2''. \tag{15}$$

Likewise, corresponding points of the nets $N_1^{(1)}$, $N_1^{(2)}$, N_{12} lie on a line, and N_{12} is a transform of $N_1^{(2)}$ by means of the function

$$\theta_{12} - \frac{\theta_1\theta_{12}''}{\theta_1''}. \tag{16}$$

By means of (11) and (12) we show that the expressions (15) and (16) are equal, and consequently the nets $N_1^{(2)}$, $N_2^{(2)}$, N_{12} form a triad.

Equation (13) is reducible to

$$x_{12} = x + [(\theta_1''\theta_2 - \theta_2''\theta_1)x' + (\theta_2'\theta_1 - \theta_1'\theta_2)x'']/(\theta_1'\theta_2'' - \theta_1''\theta_2') \tag{17}$$

From the symmetry of this expression we see that N and N_{32} are transforms of $N_1^{(1)}$ and $N_2^{(2)}$ in an analogous manner. We say that the

four nets form a *quatern* $(N, N_1^{(1)}, N_2^{(2)}, N_{12})$. This result constitutes a generalization of the theorem of permutability of transformations D_m of isothermic surfaces as established by Bianchi[2]. In like manner we have the quaterns $(N, N_1^{(2)}, N_2^{(1)}, N_{12})$ and $(N_2^{(1)}, N_1^{(1)}, N_2^{(2)}, N_1^{(2)})$. Moreover the six nets can be associated into the four triads $N, N_1^{(1)}, N_1^{(2)}$; $N, N_2^{(1)}, N_2^{(2)}$; $N_1^{(2)}, N_2^{(2)}, N_{12}$; $N_2^{(1)}, N_1^{(2)}, N_{12}$.

When the nets $N_1^{(1)}$ and $N_2^{(2)}$ have been found, the functions θ_1' and θ' are determined to within additive constants. Hence, *if* $N_1^{(1)}$ *and* $N_2^{(2)}$ *are two transforms of* N, *there exist* ∞^2 *nets* N_{12}, *each of which forms a quatern with* N, $N_1^{(1)}$ *and* $N_2^{(2)}$; *and their determination requires two quadratures.*

The six corresponding points of the nets are the vertices of a complete quadrilateral whose four sides are generic lines of the four congruences which figure in the transformations. On each of these lines there are two focal points, each being the point of contact of the line with the edge of regression of one or other of the two developables as u or v varies. The four points corresponding to the variation of either variable lie on a line, and these two lines are the tangents to the parametric curves on the envelope of the plane of the quadrilateral; moreover, these curves form a net.

Thus far we have used rectangular non-homogenous point coordinates, but in some cases it is advisable to make use of general homogenous coordinates. The four homogenous coordinates x, y, z, w, of a net satisfy an equation of the form.

$$\frac{\partial^2\theta}{\partial u \partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta. \tag{18}$$

When two nets N and N_1 are in the relation of a transformation T, the tangents to the curves $v = \text{const.}$ at corresponding points M and M_1 of the net meet in a point F_1. Likewise the tangents at M and M_1 to the curves $u = \text{const.}$ meet in a point F_2. It is readily seen that as v varies, any point on the tangent to a curve $v = \text{const.}$ of a net moves in such a way that the tangent to its path lies in the tangent plane of the net. Similarly for a point on the tangent to $u = \text{const.}$, as u varies. Since the line F_1F_2 lies in the tangent planes to both N and N_1, it is tangent to the motion of F_1 as v varies, and to the motion of F_2 as u varies. Hence F_1 and F_2 are the focal points of the congruence of lines of intersection of the planes of the nets. Following Guichard, we say that a congruence whose focal points lie on the tangents to the curves of a net and whose developables correspond to the curves of the net is *harmonic* to the net. It can readily be shown that the homogenous coordinates of a net can

be chosen so that the coordinates of the focal points of a harmonic congruence are of the respective forms

$$\frac{\partial x}{\partial u}, \quad \frac{\partial x}{\partial v}. \tag{19}$$

In this case equation (18) assumes the form (1), so that the choice of coordinates referred to its equivalent to finding a particular solution of (18).

Since the congruence F_1F_2 is harmonic to both N and N_1, it follows that the equations of any transformation T in homogenous coordinates is reducible to the form (2). If the coordinates of N satisfy (18), the equations are of the form

$$\frac{\partial x_1}{\partial u} = h\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial x_1}{\partial v} = l\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{20}$$

where now θ is a solution of (18).

When the equations of the transformation are of the form (2), each solution of the point equation of N gives a new transform by means of (5). The equations of the preceding results continue to be true, and parallel nets are replaced by any transforms.

Although these results have been stated in terms of 3-space, they hold for two dimensional spreads in n-space, provided that a congruence is defined as a two parameter family of lines possessing two families of developables.

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[2] Bianchi, *Ann. Mat., Milano*, (Ser. 3), 11, 1905, (93–158).

THE MOLECULAR WEIGHTS OF THE TRIARYLMETHYLS

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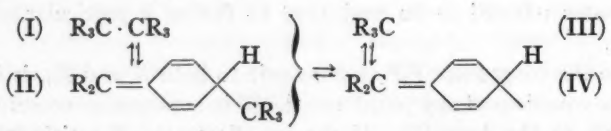
It is now generally accepted that the free radicals of the triphenylmethane series owe their unique unsaturated character to the presence of a trivalent carbon atom in the molecule. In many cases the molecular weight has been found to be double that calculated for the free radical. Nevertheless, even in these cases the presence of a compound with a single unsaturated carbon atom is still recognized, and the assumption is made that there exists, in virtue of partial dissociation, a mobile equilibrium:

$$R_3C-CR_3 \rightleftarrows R_3C+R_3C.$$

It is doubtful whether the matter is quite as simple as it is represented by the above equation. From a variety of experimental evidence[1] the conclusion seems inevitable that both the hexa-arylethanes and the triarylmethyls exist in two tautomeric modifications. It is far more likely that triarylmethyls, at least when in solution, conform to the following scheme:[2]



In other words, there is a tautomeric equilibrium between the benzenoid hexa-arylethane (I) and the quinol (II); also between the benzenoid triarylmethyl (III) wherein the central carbon atom is trivalent and its quinonoid modification (IV) wherein the *para*-carbon atom in the nucleus assumes the trivalent state. This viewpoint is in entire harmony with the peculiar chemical and physical behavior of this class of compounds. It permits us to explain, by the presence of the quinol tautomer (II), the existence of a colored modification of hexa-arylethanes in cases where molecular weight determinations fail to indicate an actually measurable degree of dissociation. And even in those cases where a partial or complete dissociation of the hexa-arylethane is demonstrable, the fact that the solution is colored finds a more reasonable and more concordant explanation in the existence of the quinonoid tautomeric triarylmethyl (IV) than merely in that of the benzenoid (III) alone.[3]

Whether, however, the supplemental hypothesis of tautomerism is accepted or not, the fact remains that we are dealing primarily with a phenomenon where a hydrocarbon dissociates spontaneously into two parts, which parts may, also spontaneously, reassociate with the production of the original substance. What are the factors which influence the degree of dissociation in a given hexa-arylethane? What is the relation between the degree of the dissociation and the nature of the aryl group in the substituted ethane?

Factors Influencing Dissociation.—Obviously, the nature of the solvent, the temperature of the solution, and the concentration of the solute must be the chief factors as regards the extent of dissociation which may be reached by a given hexa-arylethane. With regard to changes in the equilibrium between the dimolecular and the monomolecular modification due to the temperature, it has been found[4] that the molecular weight of triphenylmethyl in naphthalene at 79°–80° is approximately 414, while the average value found with other solvents such

as benzene, nitrobenzene, etc. which freeze at temperatures around 0°–5°, is between 480 and 485. A computation from these molecular weight determinations indicates that triphenylmethyl exists in the monomolecular state to an extent of 17% in naphthalene at 80°, while to considerably less than 5%, if at all, in benzene at 6°. Schlenk and Mair[5] determined the molecular weight of triphenylmethyl in benzene by the ebullioscopic method, and found that at the temperature of the boiling point of benzene, also about 80°, the hydrocarbon is dissociated in the monomolecular phase to the extent of 23.3–29.9% with approximately a 2 % concentration. Schmidlin[6] calls attention to the fact that it is very difficult to obtain an absolutely pure sample of triphenylmethyl and that samples prepared by slightly different methods often differ considerably in purity. Therefore he suggests the possibility that the variation between the molecular weights found by Schlenk with the ebullioscopic method and by Gomberg and Cone with the cryoscopic method may be due to differences in the degree of purity of the samples used. Accordingly, he constructed a special apparatus containing two thermometers, by means of which he could determine the molecular weight in benzene first by the ebullioscopic method and then by the cryoscopic method on one and the same sample. However, the molecular weights of triphenylmethyl found by the two methods showed no appreciable difference in value, and therefore no change in dissociation between the temperatures of 6° and 80° was noticeable.

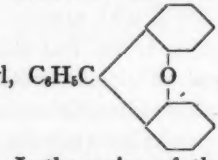
With regard to the influence of concentration upon the degree of dissociation of hexaphenylethane, an examination of the results obtained by various investigators fails to reveal a concordant influence of this factor. Neither molecular weight determinations nor the application of optical methods[7] have supplied information of a sufficiently decisive nature to warrant us in drawing any conclusion as regards the concentration influences affecting the equilibrium between the dimolecular and the monomolecular triarylmethyl.

Relation Between Dissociation and the Nature of the Aryl Groups.—Although triphenylmethyl, the first and simplest representative of the triarylmethyls, was found to be largely in the dimolecular state, some of its analogs were found to exist to a large extent as monomolecular. Among the first triarylmethyls of this kind was the series containing *p*-biphenyl groups:

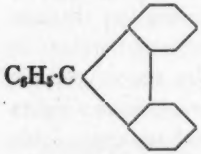
$$(C_6H_5)_2C\cdot(C_6H_4\cdot C_6H_5),\quad (C_6H_5)\,C\,(C_6H_4\cdot C_6H_5)_2,\quad C(C_6H_4\cdot C_6H_5)_3.$$

Schlenk and his co-workers concluded from the measurements of the molecular weight by the cryoscopic method with benzene as a solvent,

that these compounds are monomolecular to the extent of 15%, 80%, and 100%, respectively.[8] From these results one might infer that the dissociation of the hexa-arylethane into free radicals is greatly favored by the complexity or the weight of the aryl groups,—the dissociation becoming apparently more manifest also in proportion to the number of such groups. But the hypothesis that the dissociation of the hexa-arylethanes is proportional to the complexity of the aryl groups becomes wholly untenable when one compares triphenylmethyl with phenyl-

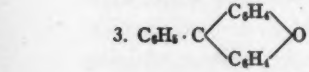
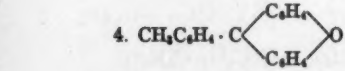
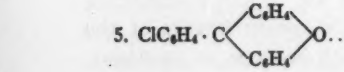
xanthyl,  , which is monomolecular to the extent of

82%.[9] Is the union of the two phenyl groups the paramount influence in this case? If so, why should a substance constitutionally so closely related to the xanthyls as phenyl-biphenylene-methyl,

 , be completely dimolecular?[10]

It is obvious that from the few facts known to us at present it is difficult, if not wholly impossible, to formulate the relation between the complexity of the aryl groups in the hexa-arylethanes on the one hand and the tendency of these ethanes toward dissociation into free radicals on the other.

The Problem of this Investigation.—In order to obtain some further knowledge as to the factors governing the equilibrium between the di- and the mono-molecular phases of triarylmethyls, we decided to select for our study a related set of compounds. The following triarylmethyls were prepared for this purpose:

1. $(C_6H_5)_3C$ ..... Triphenylmethyl.
2. $(C_6H_5)_2C \cdot C_{10}H_7$ ..... α-Naphthyl-diphenylmethyl.
3.  ..... Phenyl-xanthyl.
4.  ..... p-Tolyl-xanthyl.
5.  ..... p-Chlorphenyl-xanthyl.

6. $C_{10}H_7 \cdot C(C_6H_4)_2O$ ..... α-Naphthyl-xanthyl.

7. $C_6H_5 \cdot C(C_6H_4)(C_{10}H_6)O$ ..... Phenyl-pheno-β-naphtho-xanthyl.

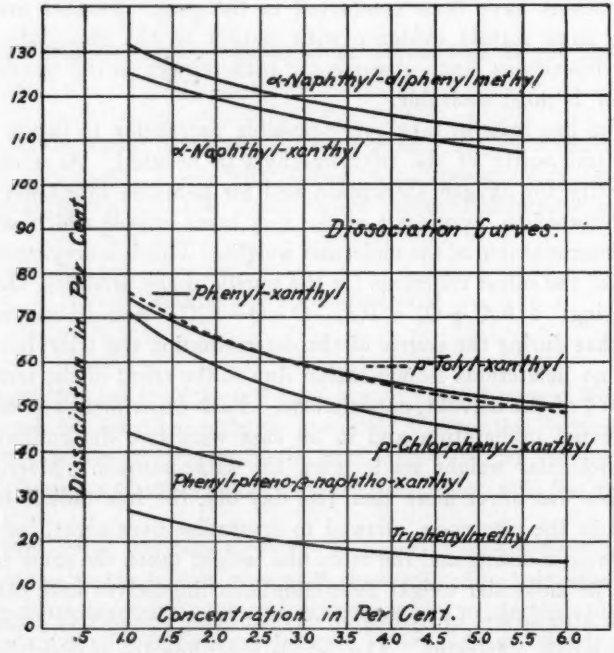
A determination of the molecular state, and consequently of the dissociation phenomenon, of these unsaturated compounds is the more pertinent in view of the limited number of radicals upon which such determinations have been conducted in the past. Further and, if possible, more certain evidence with respect to the dissociation of these hydrocarbons, and consequently with respect to the trivalency of carbon, is most desirable.

Our aim has been to take every possible precaution to insure uniformity and purity of the triarylmethyls to be used. As a check, the capacity for oxygen absorption and for peroxide formation was always resorted to, using part of the very same sample which served for the determination of the molecular weight. This is a very essential and one of the safest criterions for the purity of the products, the reaction being: $2\ R_3C + O_2 = R_3C-O-O-R_3C$. Also it was made certain that during the course of the determination the triarylmethyl suffered no deleterious isomerization due to the effect of the temperature (80°) of the solvent, naphthalene. Each experiment represents an individual preparation and in no case were two determinations of the molecular weight made upon the same sample. Moreover, the sample was never more than one day old, the free radical being prepared in the afternoon, allowed to crystallize over night, isolated the following morning and the molecular weight taken the same afternoon. The molecular weight determinations themselves were carried out in an atmosphere of hydrogen, oxygen being completely excluded from the whole apparatus. The details regarding the preparation of the pure compounds and the method used in the molecular weight determinations will appear shortly in the *Journal of the American Chemical Society*.

The triarylmethyls were studied from two points of view. (1) The molecular weight of every triarylmethyl was determined within a fairly wide range of concentration, from 1% to 6%, in order to determine the effect of concentration upon the degree of dissociation. (2) From a comparison of the degree of dissociation of various trarylmethyls, under the same conditions of solvent, concentration and temperature, it was hoped that something would be learned regarding the

influence of the various aryl groups upon the tendency of the ethane towards dissociation.

Summary of the Results.—In the diagram are represented the curves, each curve based upon a number of molecular weight determinations, expressing the dissociation tendency of the hexa-arylethanes mentioned in this paper. The curves are plotted with the percentage of concentration as abscissas and the dissociation values, in percentages, as ordinates. The values expressing dissociation were



obtained according to the formula $x = \frac{M_t - M_o}{M_o}$, where x equals the degree of dissociation, M_t represents the theoretical molecular weight of the hexa-arylethane and M_o the molecular weight actually found.

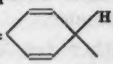
1. In all instances, a steady and gradual increase of the molecular weight is noticeable as one proceeds from a 1% concentration of the radical to that of 6%. The generality of these results, and their uniformity, leave no room for doubt that we are dealing here with a phenomenon of molecular dissociation, wherein the products of dissociation are in equilibrium with the dissociating substance. An

equilibrium of this nature might be expected to shift in favor of dissociation with dilution, as is actually the case with these compounds.

2. When in triphenylmethyl two phenyl groups become joined through an oxygen atom and thus give rise to a xanthone ring, the tendency of the new compound, phenyl-xanthyl, towards dissociation increases to a marked extent, approximately threefold.

3. A phenyl and a *p*-tolyl group are apparently equivalent in their influence for dissociation when linked to a xanthone ring; a *p*-chlorphenyl group is of somewhat lesser influence, as can be seen on comparing the dissociation curves of the three corresponding aryl-xanthyls. This equality of influence of a *p*-tolyl and a phenyl group would not necessarily have been anticipated. Tolyl-diphenylmethyl differs very much from triphenylmethyl, and tri-*p*-tolylmethyl is so unstable that it is apparently impossible to isolate it, owing to the tendency to isomerize or polymerize.

4. An α-naphthyl group when replacing a phenyl group in triphenylmethyl exerts upon the dissociation equilibrium of the compound a very decided influence in favor of the monomolecular phase—α-naphthyl-diphenylmethyl appearing as wholly dissociated. This favorable influence of the naphthyl group is still retained when the group is linked to a xanthone ring, the resulting compound being also dissociated to the extent of 100%. And yet, when the naphthyl group enters as a component in the formation of the xanthone ring itself, it depresses very decidedly the dissociation tendency of the compound, as is evident on comparing the two isomers, α-naphthyl-xanthyl and phenyl-pheno-β-naphtho-xanthyl.

5. It has been established that the triarylmethyls are, in the solid state, almost wholly devoid of color, being, like triphenylmethyl itself, only pale-yellow; but their solutions differ very much, being yellow, orange, brown, red, or green, according to the individual compound. Schlenk has described tribiphenylmethyl, 100% dissociated, as a dark green powder. Whether this difference in color from our completely dissociable radicals is real or only apparent we cannot at present say. The fact that triarylmethyls, colorless when solid, give color only when in solution, lends support to the hypothesis that the triarylmethyls do undergo tautomerization when dissolved. Not dissociation alone into triarylmethyl, but, in addition thereto, the consequent tautomerization of this into its quinonoid modification, $R_3C=$  , constitute a satisfactory explanation of the color phenomenon. Tautomerization commonly occurs, indeed,

only when the compound capable of tautomerization is in the liquid phase, or in solution. And that is just the case with the free radicals under consideration.

6. The two radicals, α-naphthyl-diphenylmethyl and α-naphthyl-xanthyl, were found to give a molecular weight *less* than that calculated for the monomolecular phase. Schlenk and Renning's[11] results show that α-naphthyl-biphenyl-phenylmethyl exhibits the same unusual behavior. We have good reasons to believe that our results are not due to experimental errors, and we took pains to verify them repeatedly. The cause of this unexpected result may possibly lie in a still further dissociation of the triarylmethyl itself, $R\,R'\,R''\,C \rightleftarrows R\,R'C + R''$.

[1] Gomberg, *Berlin, Ber. D. Chem. Ges.*, **40**, 1907, (1860); **42**, 1909, (406); Gomberg and Cone, *Liebigs Ann. Chem. Leipzig.* **370**, 1909, (190); **376**, 1910, (208).

[2] Gomberg, *Berlin, Ber. D. Chem. Ges.* (406); **46**, 1913, (228).

[3] Compare G. N. Lewis, these PROCEEDINGS, **2**, 1916,(588).

[4] Gomberg and Cone, *Berlin, Ber. D. Chem. Ges.*, **37**, 1904, (2037).

[5] Schlenk and Mair, *Liebigs Ann. Chem., Leipzig*, **394**, 1912, (179).

[6] Schmidlin, *Das Triphenylemethyl*, Stuttgart, 1914, (94).

[7] Piccard, *Liebigs Ann. Chem., Leipzig*, **381**, 1911, (347).

[8] Schlenk, *Ibid.*, **372**, 1909, (4); **394**, 1912, (186); *Berlin, Ber. D. chem. Ges.*, **43**, 1910, (1756); Schmidlin and Garcia-Banus, *Ibid.*, **45**, 1912, (3176).

[9] Schlenk and Renning, *Liebigs, Ann. Chem, Leipzig*, **394**, 1912, (189).

[10] Schlenk, Herzenstein and Weickel, *Berlin, Ber. D. chem. Ges.*, **43**, 1910, (1754).

[11] Schlenk and Renning, *Liebigs Ann. Chem., Leipzig*, **394**, 1912, (195).

SEX-DETERMINATION AND SEX-DIFFERENTIATION IN MAMMALS

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The principle of zygotic sex-determination is generally regarded as established for mammals as for other animal groups. The reasons for this are (1) the identity of sex of all individuals derived from a single zygote: e.g., identical twins, quadruplets of armadillos, etc.; (2) the facts of sex-linked inheritance, which demonstrate the inheritance of certain sex factors in a Mendelian way; (3) the dimorphism of spermatozoa in mammals as in other groups with zygotic determination of sex. We must therefore regard sex as *determined*, in the usual sense of the word, at the time of union of the gametes.

The question, however, arises, whether sex-*determination* involves an irreversible tendency to the corresponding sex-*differentiation*, or

whether such differentiation is more or less controllable, or even completely reversible?

Up to a certain stage in the development of mammals there are no morphological evidences of the determined sex. Prior to this stage sex-characters are identical in both kinds of zygotes; the gonad first enters on a phase of male differentiation in both sexes, which subsequently changes to the female direction in the female zygotes only; both male and female sex-ducts arise in each kind of zygote, and by subsequent corelative progressive and retrogressive differentiation the conditions of the appropriate sex are produced; external parts also appear similarly at first in both sexes. The possibilities for complete reversal of the indicated sex-differentiation would therefore seem to lie within this so-called sexually indifferent stage, and to diminish progressively as differentiation proceeds.

There are many indi ations that each zygote, whether determined as male or female, has both tendencies, both sets of sex-factors; in other words that the reactions for male or for female differentiation are both possible for each zygote; but that they are to a considerable extent mutually exclusive in bisexual animals. The initial sex-determination is, therefore, a condition in which there is a quantitative superiority of one or the other tendency or set of factors. The advance of development progressively limits the possible operations of the inferior set of factors, so that, by both positive and negative limitations, reversal of the initial sex-index becomes increasingly more difficult.

It has long been known that the degree of development of the sex-characters that arise after birth is dependent in mammals upon internal secretions of the sex-glands (sex-hormones) circulating in the blood. This is seen in the well-known effects of castration; and partial reversal of sex-differentiation has been secured by implantation of the sex-gland of the opposite sex following castration.[1] But the more fundamental sex-characters, like other fundamental characters, are differentiated in mammals before birth. Such are the type of sex-gland, whether ovary or testis, the type of sex-ducts, whether vasa deferentia or female reproductive tract, and the type of the external organs of reproduction.

The problem of the extent to which sex-differentiation may be reversible carries us back therefore to the sexually indifferent stage, and the question arises whether the sex-characters that develop before birth are, like those arising after, dependent for the degree of their development upon sex-hormones, and whether, like the latter also, they are more or less reversible by action of the sex-hormones of the opposite sex?

The investigation of a remarkable phenomenon in twins of cattle furnishes a positive answer to this question so far as the female is concerned.

The female of two-sexed twins in cattle, commonly known as the free-martin, has long been known to be absolutely sterile as a general rule; however a small percentage of such females is perfectly normal. I have found by a study of the embryonic development that the phenomenon of sterility is due to fusion of the embryonic membranes of the twins, and an anastomosis of the arteries and veins of the female and male associates, but more especially of the arteries, so that there is literal community of blood during foetal life.[2] If the anastomosis of the blood-vessels does not take place, the female is perfectly normal as is usual in the twins or multiple births of all other mammals. In cattle again if the twins be of the same sex both are perfectly normal.

Nature has thus performed here a perfectly controlled experiment, which shows that blood community of foetal life between embryos of different sex causes sterility of the female. This fact can be explained only on the assumption that the foetal blood carries specific sex-hormones, because the only system of the female that is affected is the reproductive system. The male, on the other hand, is normal in all its parts, and this finds explanation in the fact that the sexual differentiation of the male antedates by a little that of the female, and the development of female sex-hormones is probably inhibited from the start.

The time at which the anastomosis of the blood vessels occurs is a question of fundamental importance. Sex-differentiation begins in cattle when the embryo is about 25 mm. long. The evidence at my command indicates strongly that fusion of the embryonic membranes begins at about this time or a little earlier; in a pair of twins 15 mm. long the membranes overlapped but were not yet fused; in another pair, of which one member was 35 mm. and the other 40 mm. in length, the membranes were perfectly fused so that the place of union could no longer be detected. The vascular areas of the two sides overlapped, but the larger vessels did not anastomose; a capillary connection at least between the two sides certainly existed. In another pair 50 mm. long the fusion was perfect and the anastomosis of the blood-vessels also. Study of normal embryos also shows that the conditions precedent to fusion are fully established before the 20 mm. stage; and study of the gonads of the free-martin shows that the development of the ovarian cortex is probably inhibited from the beginning, i.e., from about the 25 mm. stage. There can be no doubt that the blood community dates from about the time of the sexually indifferent stage, but it certainly varies more or less

both with reference to the time of origin and the extent of the vascular anastomosis.

We have hitherto noted only that the free-martin is sterile whenever blood-community with its male twin exists during foetal life, i.e., in about seven-eighths of all cases, and that otherwise it is normal. What is the nature of this sterility? Observers from the time of John Hunter[3] (1786), who have studied the anatomy of free-martins, have all noted the intersexual character of its reproductive system—the internal organs of reproduction are largely male in type, the external female—and the later students of the free-martin have generally regarded it as a modified male on account of the character of the internal organs and for other reasons (Spiegelberg,[4] D. Berry Hart,[5] Bateson,[6] Cole[7]). The gonad is sometimes absent or exceedingly rudimentary, but when well developed it never exhibits any trace of ovarian cortex, and its structure is testis-like, though germ-cells are not formed. The sexual ducts show reduction or absence of the female parts and a graded series of development of the vas deferens. The external parts are usually typically female. We have on the one hand, therefore, failure of development of the internal female reproductive organs, and, on the other hand, the male parts, which usually degenerate, undergo development. Anatomically the free-martin is definitely intersexual, to a variable extent as will be seen.

The writer has studied the anatomy of 22 foetal free-martins ranging in size from 7.5 to 28 cm. The striking results of this examination were (1) The gonads remain rudimentary in size during this period; (2) the female ducts fail to develop; they frequently remain in part as undeveloped rudiments, but in other cases disappear as completely as in the male. (3) The male ducts develop in varying degrees, always more than in the normal female, though rarely to the same extent as in the male. (4) Gubernacula invariably developed in the free-martins and formed peritoneal evaginations exactly as in males. (5) In one of the oldest foetuses the gonads had entered the saccus vaginalis as in the male. (6) In my material the external organs were always typically female.

Miss Chapin[8] made a histological investigation of the gonads of the foetal free-martin under my direction and determined that the ovarian cortex never forms, but that a quite typical albuginea develops over the surface as in the testis of the male. The medullary cords, homologue of the seminiferous tubules of the male, also underwent unusually great development.

A seven weeks old free-martin examined by the writer possessed testes

and vasa deferentia, but no trace of uterus or vagina. The testes had descended and lay beneath the skin in the region of the groin. No scrotum was formed and the external parts were typically female. A free-martin described by Numan in 1844 was even farther transformed in the male direction, and the external parts were also modified.

The various cases can be arranged in a series of increasing male-likeness, but the transformation of the female zygote owing to action of the male sex-hormones does not in the case of the free-martin, in the material at our command, proceed all the way to the normal male condition. It is perhaps worth noting that, if it ever did, we would be unable to detect it, except on a basis of much larger statistics than we possess. But the rarity of the more extreme detectable cases makes it seem very improbable that other cases jump all the way across the gap to the normal male.

It follows from the data that the female zygote must contain factors for both sexes; the primary determination of the female sex must therefore be due to dominance of the female factors over the male. If we think of this as a simple quantitative relation, as Goldschmidt[9] (1916) has done, we can explain the intersexual condition of the free-martin as due to an acceleration or intensification of the male factors of the female zygote by the male hormones. The degree of the effect which is quite variable, as we have seen, would of course be subject to all quantitative variations of the hormone. Thus the case of the free-martin could come under the same general point of view as that of the intersexes of Lymantria according to Goldschmidt with the one exception that the quantitative differences between the male and female factors of the female zygote necessary for the differentiation of female characters, are reduced in the free-martin by internal secretions instead of by variations of potency of the male factors in different varieties as in the intersexual hybrids of Lymantria.

The case of the free-martin shows that a gonad with a primary female determination may form a structure which is morphologically a testis, (cf. Chapin 1917) through suppression of the cortex and over development of the medullary cords and urinogenital union under the influence of male sex-hormones. Lesser degrees of transformation are of course possible, so that it is certain that the gonad of a mammalian female zygote is capable of most, at least, of the series of transformations that may exist between an ovary and a testis. Whether the transformation in the male direction may proceed under such conditions to the production of true spermatocytes and spermatozoa is at least doubtful. Such elements have not hitherto been described for free-martins, if we

except D. Berry Hart's statement concerning the gonads of Hunter free-martins, that "in only one are spermatozoa present." More than six words seem necessary to establish so important an exception.

Regarding other parts of the internal reproductive system we have seen that the free-martins exhibit a graded series of inhibition of the female ducts and of development of the male ducts which may obviously correspond to variable time of onset, intensity, and perhaps duration of action, on the male sex-hormones. The series extends nearly to the normal male limit in exceptional cases. There is indicated a rough parallelism at least between the grade of transformation of the gonad and that of the remainder of the internal reproductive system. The external organs of reproduction are the least liable to modification, but they do not escape in all cases, and may even exhibit considerable transformation in the male direction, if we can accept Numan's case.

The fundamental determining factor in these events is undoubtedly the male sex-hormones as has been argued previously, but the entire causal nexus is by no means clear. We do not know what the results of embryonic castration of the female might be in itself, and hence we are unable to assert definitely in just what positive ways the male hormones act on the female zygote, because the earliest determinable result of such action is the suppression of the ovarian cortex, which must be regarded as practically equivalent to castration. This action at least is due to the male hormones; how much of the subsequent events is due to mere absence of ovarian tissue, and how much to positive action of male sex-hormones is more or less problematical. It is well known that spayed females of certain birds and mammals tend to develop male characters; heifers with cystic degeneration of the ovary also develop certain male characteristics (Pearl and Surface, 1915),[10] so that we must admit in principle the possibility that much of the male development in the free-martin is due to the lack of inhibitions normally furnished by the ovary.

It is also probable that the various parts of the reproductive system have other means of correlation, and act and react on one another in various ways. Certain indications of this are seen in lateral variations, as for instance in one of my cases where a large gonad on one side is associated with a large Wolffian duct, and seminal vesicle, and a much smaller one on the other side with a correspondingly smaller duct and vesicle.

When, therefore, we attribute the free-martin condition to the male hormones we only mean to assert that they are the primary cause, and not that they are the decisive factors in each member of the series of events.

The possibility exists, however, that definitely planned experiments may enable us to regulate time and dosage of hormones better than is done in this experiment of nature; the results of such experiments cannot of course be foreseen. Nor can it be predicted in advance what the results of the inverse experiment might prove to be, i.e., treatment of the male zygote from the beginning of six-differentiation with female hormones. Such experiments will be necessary for the full solution of the stated problem. We can, however, state, confidently on the basis of the present results that sex-determination in mammals is not irreversible predestination, and that with known methods and principles of physiology we can investigate the possible range of reversibility.[11]

[1] Steinach, E., *Zentralbl. Physiol.*, B., **27**, 1913, (717–723).

[2] Lillie, F. R., *Science, New York*, N. S., **43**, 1916, (611–613).

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THE CRYSTAL STRUCTURE OF MAGNESIUM

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Magnesium is assigned by crystallographers to the holohedral class of the hexagonal system, with axial ratio 1.624. The structure given below agrees with this symmetry. The arrangement of atoms is that of hexagonal close packing, the arrangement which equal hard spheres assume when closely packed, except that the structure is shortened by about one-half of one percent in the direction of the hexagonal axis. This is the fourth type of arrangement of atoms in elementary substances thus far observed, viz.: the diamond type, characteristic of diamond, silicon, bismuth and antimony, where each atom is surrounded by four equidistant nearest neighbors; the centered cubic lattice, characteristic of iron, and probably also of nickel and the alkali metals, where each atom is surrounded by eight equidistant nearest neighbors; the face-centered cubic lattice, or closest-packed cubic arrangement,

of which aluminum, copper, silver, and probably gold and lead are examples, where each atom has twelve equidistant nearest neighbors; and the closest-packed hexagonal arrangement described below, of which magnesium is at present the only example, where the number of equidistant nearest neighbors is also twelve, but in a slightly different arrangement from those of the face-centered cubic lattice.

The X-ray analysis was made in two parts. First, single small crystals were mounted with definite orientation on the spectrometer table, and photographed while slowly rotated and exposed to a monochromatic beam of X-rays. This gave the approximate structure. A picture was then taken of magnesium powder, in the manner described in a previous paper (*Phys. Rev.* **9**, 85, Jan. 1917), which checked and confirmed the results of the first method.

Three small samples, formed by vacuum distillation, were used for the single photographs. The first was mounted with its basal plane (0001) parallel to the rays, and was rotated about the axis (0001) – ($10\bar{1}0$) for about 30° on each side of the center. This should give reflection from (0001) and the flatter pyramids ($10\bar{1}3$), ($10\bar{1}2$), ($10\bar{1}1$), etc. The second was mounted so as to rotate about the same axis, but with $10\bar{1}0$ parallel to the rays at the center position. This should give reflection from ($10\bar{1}0$) and the steeper pyramids ($30\bar{3}1$), ($20\bar{2}1$), ($10\bar{1}1$), etc. The third was rotated about the axis (0001), (1120), with rays parallel to $11\bar{2}0$ at center, so as to give reflection from ($11\bar{2}0$), ($11\bar{2}1$), ($11\bar{2}2$), etc.

The observed lines and spacings are given in table 1.

TABLE 1

CRYSTAL 1			CRYSTAL 2			CRYSTAL 3		
Position of line	Spacing of plane	Plane	Position of line	Spacing of plane	Plane	Position of line	Spacing of plane	Plane
3.10	2.59	0001	2.90	2.75	$10\bar{1}0$	5.02	1.60	$11\bar{2}0$
2.90	2.75	$10\bar{1}0$	3.10	2.59	0001	5.50	1.48	$11\bar{2}1$
3.30	2.44	$10\bar{1}1$	3.30	2.44	$10\bar{1}1$	6.0	1.36	$11\bar{2}2$
4.20	1.90	$10\bar{1}2$	6.05	1.34	$20\bar{2}1$			
5.50	1.48	$10\bar{1}3$	8.92	0.92	$10\bar{1}3(3)$			
6.27	1.30	$20\bar{2}1$	5.9	1.38	$10\bar{1}0(2)$			

The first column gives for each crystal the distance of the observed line from the center, the second the spacing of the corresponding plane, as calculated from this distance, and the third the indices of the plane.

A triangular prism having the spacing $d_{11\bar{2}0} = 1.61$ A and axial ratio 1.624 would have a height $1.624 \times 2d_{11\bar{2}0} = 5.23$ A, which is exactly twice the spacing of the (0001) planes found above, and suggests that the lattice is composed of *two* sets of triangular prisms each of side 3.22A

and height 5.23A, the atoms of either set being in the center of the prisms of the other. The number n of atoms per unit prism is, if ρ is the density, M the mass of an atom of Mg, and a and h represent the side and height of the prism, respectively:

$$n = \frac{\sqrt{3}}{4} \frac{a^2 h \rho}{M} = 1.03$$

which is equal to 1 within the limit of accuracy of a, h, and ρ, and is correct for the assumed structure.

The second step in the analysis was to check this assumed structure by a photograph taken through finely powdered magnesium, which should show all the lines required by the assumed structure *and no more*. Table 2 gives the position of the observed lines and the corresponding spacing

TABLE 2

DISTANCE OF LINE FROM CENTER	ANGLE OF REFLECTION	INTENSITY (ESTIMATED)	SPACING OF PLANE IN ANGSTROMS		INDICES OF PLANE
			Experimental	Theoretical	
cm.					
2.92	7.40°	70.	2.75	2.75	$10\bar{1}0$
				2.59	0001
3.30	8.33	150.0	2.44	2.44	$10\bar{1}1$
4.23	10.74	50.0	1.91	1.90	$10\bar{1}2$
5.03	12.75	70.0	1.61	1.60	$11\bar{2}0$
5.50	13.90	60.0	1.48	1.48	$10\bar{1}3$
				1.38	$10\bar{1}0$ (2)
5.95	15.00	60.0	1.36	1.36	$11\bar{2}2$
6.05	15.30	20.0	1.34	1.34	$20\bar{2}1$
				1.30	0001 (2)
6.65	16.80	10.0	1.23	1.23	$10\bar{1}1$ (2)
6.90	17.40	0.5	1.18	1.18	$10\bar{1}4$
7.52	19.00	15.0	1.09	1.08	$20\bar{2}3$
7.96	20.10	30.0	1.04	1.05	$21\bar{3}0$
8.10	20.50	0.5	1.02	1.03	$21\bar{3}1$
				1.01	$11\bar{2}4$
8.41	21.3	20.0	0.98	0.97	$21\bar{3}2$
					$10\bar{1}5$
				0.94	$10\bar{1}2$ (2)
8.83	22.4	0.3	0.93	0.92	$10\bar{1}0$ (3)
9.15	23.1	15.0	0.90	0.89	$21\bar{3}3$
9.48	24.0	10.0	0.87	0.87	$30\bar{3}2$
					0001 (3)
9.90	25.1	0.5	0.83	0.83	$10\bar{1}6$
					$20\bar{2}5$
				0.82	$10\bar{1}1$ (3)
					$21\bar{3}4$
				0.80	$11\bar{2}0$ (2)
10.95	27.7	10.0	0.77	0.77	$31\bar{4}0$

of the planes, together with the theoretical spacing for the lattice described above. The agreement is within the limit of error of the measurements, except that some of the predicted lines are too faint to show. This is to be accounted for by the distribution of electrons in the atoms, and will be discussed in a future paper.

THE STRUCTURE OF HIGH-STANDING ATOLLS

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The structure of high-standing atolls has seldom been studied in detail, and is perhaps seldom sufficiently revealed for close study. Attention is therefore drawn here to only one structural feature, namely the relation of atoll limestones to their supposed foundation of volcanic rocks. According to Darwin's theory of intermittent subsidence, the limestones of atolls should lie unconformably on an unevenly eroded, submountainous volcanic mass, the top of which may be buried to any depth, as in section M of sector L, figure 1: the section of the volcanic

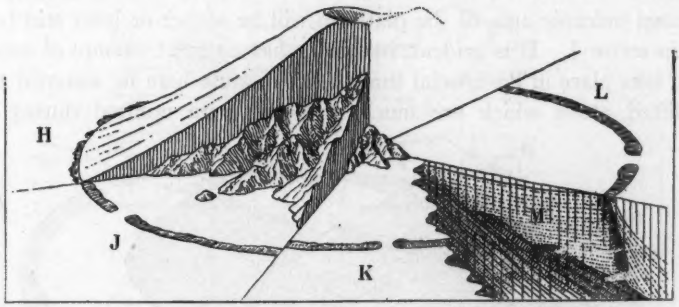

FIG. 1.

foundation here shown resulting from the dissection and progressive subsidence of a volcanic cone, as shown in sectors H, J, K. According to the Glacial-control theory, which is today the only fully formulated competitor of Darwin's theory that deserves consideration here, the limestones of atolls should as a rule unconformably overlie a flat platform of volcanic and calcareous rocks, produced by the following processes: A preglacial volcanic island, sector A, figure 2, is supposed to have stood still so long as to have been worn down to low relief, as in sector B, while a reef plain was built by outgrowth around it: during the Glacial period, when the ocean was lowered about 40 fathoms and

chilled enough to kill the reef-building corals, the island was progressively cut away by the sea, sectors C, D, and eventually reduced to a platform a little below the sea surface, sector E: when the sea rose and warmed in Postglacial time, a reef was built up around the platform margin, and the enclosed area was covered with lagoon deposits, the resulting structure being shown in section on the face of sector F.

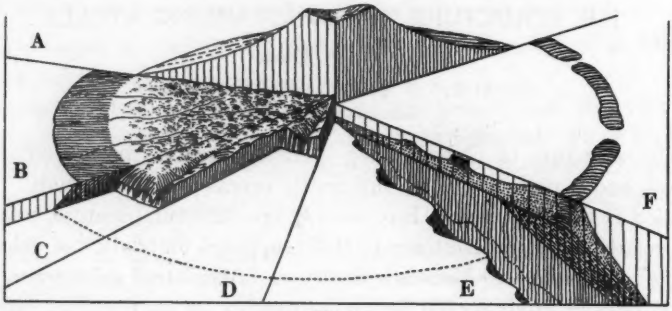

FIG. 2.

Now if such an atoll, sector F, figure 3, be uplifted more than 40 fathoms or 240 feet, as in sector G, and eroded, as in sector H, the central volcanic area of the platform will be sooner or later laid bare, as in sector J. It is evident, however, that no great amount of erosion can take place in Postglacial time; hence it must here be assumed that uplifted atolls which are much dissected were uplifted during the

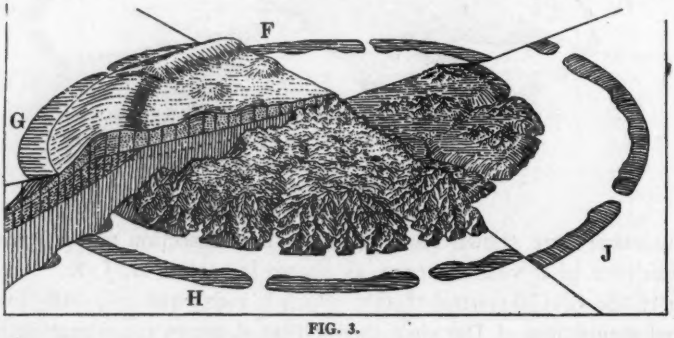

FIG. 3.

Glacial period, and then deeply dissected by subaerial erosion and benched by marginal abrasion during the last Glacial epoch of lowered sea level, so that when the sea finally rose in Postglacial time, an outstanding barrier reef would grow up around the margin of the last abraded marginal bench and enclose a lagoon, the waters of which would enter narrow embayments in the central island, as in sector H. In

case an island were dissected sufficiently to show its abraded platform of central volcanic rocks surmounted by residual limestone hills, as in sector J, a still earlier uplift would be demanded; and in such case, a correspondingly shorter part of the Glacial period would be allowed for the initial abrasion of the volcanic island. The more clearly these various consequences of the theory are conceived, the more closely can the theory be tested when the consequences are confronted with the facts, to which we may now turn.

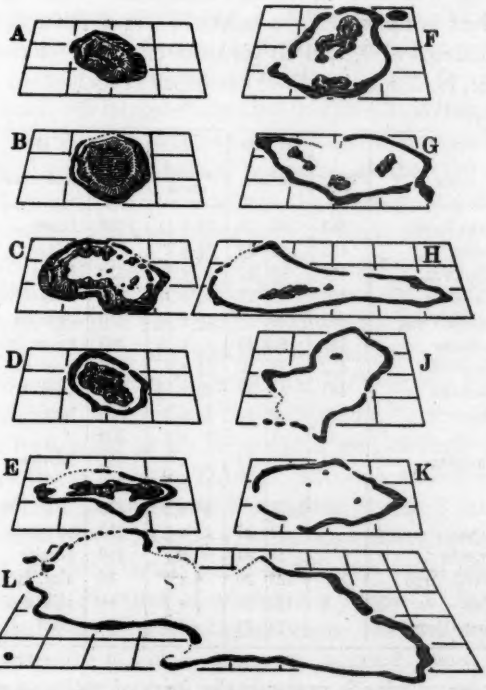

FIG. 4.

A number of uplifted and dissected atolls occur in the Lau group of southeastern Fiji. The route of my Shaler Memorial voyage of 1914 did not, to my regret, lead me to them, but most of them have been described in some detail by Gardiner[1] and Agassiz,[2] from whose reports the following items are taken. The rough outlines of figure 4 are constructed in perspective, with exaggerated height, from Agassiz charts; the squares in the perspective network are 2 nautical miles on a side. The dimensions of the uplifted atolls and of some neighboring sea-

level atolls are presented in the table below, in which the first column gives the letter by which the island, named in the second column, is designated in figure 4; the third column gives the page in Gardiner's report, and the next two columns the page and the plate in Agassiz' report, where descriptions and charts of the atolls may be found. The table is arranged with the best preserved atolls, adjoined by sea-level fringing reefs, as its first members; with dissected limestone islands enclosed by barrier reefs as its middle members; and with two almost-atolls and one true atoll as its final members. Columns 6 and 7 give the dimensions in miles and the height in feet of the uplifted calcareous islands. Columns 8, 9, and 10 give the character of the surrounding sea-level reef, its dimensions, and the depth of its lagoon.

(1) LETTER	(2) NAME	(3) GARDINER	(4) (5) AGASSIZ		(6) DIMENSIONS	(7) HEIGHT	(8) NEW REEF	(9) DIMENSIONS	(10) DEPTH
	Vatu Varâ	462	53	19	$1\frac{1}{2} \times 1\frac{1}{2}$	1030	fringe	$1\frac{1}{2} \times 2$	
A	Naiau	462	52	20	$3\frac{1}{2} \times 2$	580	fringe	$4 \times 2\frac{1}{2}$	
B	Kambara	463	98	22	3×5	320	fringe	$3\frac{1}{2} \times 5\frac{1}{2}$	2
	Wangava	461	66	22	$1\frac{1}{2} \times 3$	290	fringe	$4\frac{1}{2} \times 2$	8
	Vanua Vatu	462	121	21	$1\frac{1}{2} \times 1\frac{1}{2}$	310	close br.	$2\frac{1}{2} \times 3$	2
C	Fulanga	457	62	23	$5 \times 3\frac{1}{2}$	260	fringe-br.	$5\frac{1}{2} \times 4\frac{1}{2}$	4–5
D	Tuvuthá	462	51	20	$3\frac{1}{2} \times 2$	800	close br.	$4\frac{1}{2} \times 3$	8–9
E	Namuka	461	57	22	$4\frac{1}{2} \times 1\frac{1}{2}$	260	barrier	$7\frac{1}{2} \times 2\frac{1}{2}$	11–13
F	Ongea	460	60	22	4×2	270	barrier	5×8	10–13
					1×2	300			
G	Yangasa	461	57	22	$2 \times \frac{1}{2}$	390	barrier	$9 \times 5\frac{1}{2}$	16–19
					$1 \times \frac{1}{2}$	210			
					$\frac{1}{2} \times \frac{1}{4}$	270			
	Aiwa		54	21	$2 \times \frac{1}{4}$	210	barrier	9×3	18–23
H	Oneata		56	21	$3 \times \frac{3}{4}$	160	barrier	11×2–5	18–20
J	North Argo		125	20	$\frac{1}{2} \times \frac{1}{8}$	80	alm. atoll	5×9	18–21
K	Reid		124	20	$\frac{1}{4} \times \frac{1}{8}$	50	alm. atoll	7×6	18–21
L	Great Argo		124	21			atoll	22×9	30–36

The first seven islands preserve the form of atolls so well that their emergence must be of recent, postglacial date, and may have been nearly synchronous; but as their altitudes vary greatly, uneven uplift and not a fall of ocean level must be appealed to in accounting for their emergence. It is noteworthy that all these little-dissected islands are surrounded by sea-level reefs of the fringing or close-set barrier type. On the other hand, the five following islands, which lie to the eastward of the preceding seven, do not present the form of atolls; they are of irregular outline in plan and profile; two of them are discontinuous groups of small limestone knobs. If any other origin than uplifted

atolls were available for high-standing oceanic limestone islands, these examples might be ascribed to it; but in the absence of other origin, it seems reasonable to regard them as maturely dissected atolls. They must have been uplifted earlier than their little-dissected neighbors, probably during the Glacial period and not after its close. It is here noteworthy that the surrounding sea-level reefs are relatively distant barriers. Two other examples, North Argo and Reed, take their names, not from the little islands that they enclose, but from the enclosing reefs themselves: they lie farther east than the others and are classed today as almost-atolls; one of the islands in the North Argo lagoon is "stated to be of volcanic origin;" both islands in the Reid lagoon are "probably of elevated limestone." Finally, Great Argo reef is a true atoll, the largest of its kind in Fiji. It is noteworthy that these islands are so distributed as to indicate a westward wave-like progression of a meridional belt of upheaval, so that the earliest uplifted reefs, all lying to the eastward, are at present greatly dissected and somewhat depressed, while the latest uplifted reefs, all lying to the westward, are little dissected; farther west still a number of islands show no signs of uplift, as if the wave of uplift had not yet reached them.

Now although the first ten islands of the table are higher than 240 feet, they do not exhibit any volcanic foundation. The first five of these are, however, so little dissected that the volcanic foundation, if above sea-level, may be concealed by its limestone cover. The absence of a volcanic foundation in the next five argues strongly against the Glacial-control theory; for in Fulanga, although the reef rim is not much dissected, the enclosed area has depths of 4 or 5 fathoms in its lagoon, the bottom of which is thus 290 feet below the rim crest; Tuvuthá, which reaches the exceptional height of 800 feet in its northern part, reveals no volcanic platform at or below a height of 500 feet in its center; Namuka, Ongea, and Yangasa, more dissected and embayed than the hypothetical island of sector H, figure 3, show no volcanic platform, although the vertical measure from their somewhat aggraded lagoon floors to their somewhat worn-down summits ranges from 330 to 500 feet; this is amply sufficient to reveal a volcanic platform if it occurred at the depth of 240 feet below the original reef-level of the now dissected islands. Aiwa and Oneata, small limestone islands in rather large lagoons, although they have undoubtedly lost something of their original height, have vertical measures of 350 and 280 feet between lagoon floor and island top. The three remaining examples are admirable illustrations of Agassiz' theory that some atolls are derived from uplifted and worn-down limestone islands; but the sequence of

forms here given does not, to my reading, support his view that the uplifted limestone islands were not atolls of an earlier generation. Indeed it is a good deal of an assumption that Great Argo reef, a true atoll today, has ever been uplifted, for it contains no limestone islands: the reason for supposing it to have been uplifted is, that limestone islands, mere remnants of formerly larger masses, occur inside of the neighboring barrier reefs. But if Great Argo reef represents an uplifted and worn-down atoll, its uplift must have been relatively early because its erosion is completed; and if its uplift were early, its previous abrasion must according to the Glacial-control theory, have been accomplished in much less than the whole of the Glacial period; yet this is the largest atoll in Fiji.

A sixteenth example might be added, farther north than the others and about on the meridian of the maturely dissected limestone islands, although in its original form before uplift it appears to have been not a true atoll, but an almost-atoll: that is, a reef enclosing a lagoon in which a small volcanic island still survived. This is the group of islands, of which Vanua Mbalavu is the largest, enclosed by the great Exploring reef, some account of which has been given in an earlier article (these PROCEEDINGS, **2**, 1916, 471–475). The original sea-level outline of this almost-atoll reef appears to have enclosed a large and irregular lagoon, 15 by 25 miles in diameter, in the western part of which a small volcanic ridge rose in Pliocene or Pleistocene time to a height of 200 or 300 feet: after an uplift of over 600 feet, the limestone plateau was greatly eroded, and reduced for the most part to moderate or small relief, so that in Pleistocene time its larger and higher surviving fragments, partly limestone, partly volcanic, were but a small fraction of the original mass; then the resulting lowland was submerged, and the present barrier reef was built up around its margin; but be it noted that this recent submergence cannot be fully accounted for by the Postglacial rise of ocean level, because the enclosed lagoon floor deepens from 20 fathoms near its western side to over 100 fathoms at its eastern side, thus implying a recent tilting, as Agassiz noted; and this tilting would represent the sinking side of the wave-like upheaval above mentioned. The surviving islands are pertinent in the present connection because several of them show volcanic rocks unconformably covered by eroded limestones, remnants of the uplifted almost-atoll: the contact of the two kinds of rock is not a level platform at a depth of about 240 feet below the highest limestones; on the contrary, the contact exhibits rounded forms and moderate slopes such as characterize volcanic islands maturely dissected by subaerial

erosion; and the occurrence of such forms beneath heavy limestones, 600 feet or more in thickness, clearly demonstrates the submergence of a previously eroded volcanic mass by over 600 feet, while the limestones were forming. Thus not only the recent history of the present barrier reef around Vanua Mbalavu, but also the Pleistocene history of the now dissected almost-atoll, of which Vanua Mbalavu is a remnant, testifies unqualifiedly in favor of Darwin's theory of coral reef and against all other theories. [Since the above was written Foye[3] gives independent evidence of the eastward tilting of Lakemba, which is on about the same meridian as Vanua Mbalavu.]

[1] J. S. Gardiner, *Cambridge, Eng., Proc. Phil. Soc.*, **9**, 1898, (417–503).
[2] A. Agassiz, *Cambridge, Mass., Bull. Mus. Comp. Zool., Harvard Coll.* **33**, 1899, (1–167).
[3] W. G. Foye, *Amer. J. Sci., New Haven*, **43**, 1917, (343–350).

STUDIES OF MAGNITUDE IN STAR CLUSTERS, VII. A METHOD FOR THE DETERMINATION OF THE RELATIVE DISTANCES OF GLOBULAR CLUSTERS

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More than 150 variables for which the light changes are rapid and periodic have been found among the thousand brightest stars in the globular cluster Messier 3. Eighty per cent of them were discovered twenty years ago by Professor Bailey at Harvard,[1] and the remainder three years ago by the writer at Mount Wilson.[2] The light variations of these stars are typical of a large class of variables—the short period Cepheids—some of which are found among the stars in the sky at large, though the far greater majority of those now on record are confined to a few of the globular clusters and to the Magellanic clouds. Wherever found they appear remarkably alike in range of variation, spectral type, color variation, length of period, nature of light changes, and even in the irregularities of the periods and the fluctuations of the light curves.

Recent work with the 60-inch reflector on the variables in Messier 3 is supplemental to the determination of light curves and periods by Bailey,[3] and is incorporated in the general study of magnitudes in clusters primarily for the intercomparison, on the basis of the Mount Wilson scale of magnitudes, of the brightness of variables in this and other globular systems. It is part of an investigation of the magnitudes and colors of all the brighter stars in Messier 3, and follows the methods previously employed.[4]

As much of Bailey's work preceded the establishment of the North Polar Sequence, his values of the magnitudes and ranges of variation may now be standardized. A sample of the revised data is given in table 1. The variables differ very little from each other in any respect, and particularly significant is the striking similarity of the median magnitudes. Bailey derived light curves and periods for 110 variables in this cluster, and 54 of them were chosen as fairly free from uncertainty. The median magnitude (defined as the mean of maximum and minimum) of these 54 stars is 15.49, on the Mount Wilson system, with a probable error of less than a hundredth of a magnitude. The average deviation from this mean for a single star is ± 0.07, and the largest deviation is less than two-tenths of a magnitude. If all 110 variables are used, the mean is 15.50 ± 0.006, and the average deviation is ± 0.08. The distribution of the residuals follows the law of error as closely as could be expected for a small number of values.

TABLE 1

STAR	PERIOD	BAILEY'S MAGNITUDE		MOUNT WILSON PHOTOGRAPHIC MAGNITUDE			MEDIAN MAGNITUDE
		Maximum	Minimum	Maximum	Minimum	Range	
	days						
9	0.542	14.60	16.76	14.82	16.15	1.33	15.48
10	0.570	14.75	16.70	14.92	16.11	1.19	15.52
27	0.580	15.00	16.68	15.06	16.10	1.04	15.58
34	0.559	15.00	16.70	15.06	16.11	1.05	15.58
38	0.561	14.80	16.83	14.95	16.19	1.24	15.57
40	0.552	14.70	16.83	14.88	16.19	1.31	15.54
49	0.548	14.76	16.76	14.93	16.15	1.22	15.54
63	0.570	14.80	16.62	14.95	16.06	1.11	15.50
80	0.539	14.74	16.70	14.91	16.11	1.20	15.51

Great extremes of absolute brightness are known to exist in Messier 3—nine or ten magnitudes, at least. More than 20,000 stars fainter than these variables have been photographed at Mount Wilson, but no other variables of this or other types have been found among them. The result is confirmed by an examination of long-exposure plates at Harvard. The periodic light variations are apparently confined to a narrow interval of brightness.

Since the deviations of the median magnitudes from their mean are far within the errors of observation, the conclusion is forced upon us that in Messier 3 short-period variation is associated with stars of a very definite intrinsic luminosity. The situation of the variables in a distant globular system necessitates the equality of absolute as well as of apparent magnitudes. Moreover, the work on color, so far as it has

gone, implies that the spectra are all likewise strictly comparable. We conclude, therefore, that in surface brightness and volume, and probably in mass, density, and other physical properties, these 110 stars are almost identical.

Whether the isolated variables of this so-called cluster type also are exactly alike in luminosity, we have at present no means of knowing, other than analogy, because their distances are unknown and their apparent magnitudes differ greatly. To investigate further the possible generality of a law of constant median magnitude, some special studies of the variables in other clusters have been made. The extended discussion will appear in the *Astrophysical Journal*. The results are summarized in the following numbered paragraphs.

1. In Messier 5 Bailey has recently determined the periods and light curves of about 70 variables.[5] Excluding those that are nearer the center than 1'.2, the results for which must be uncertain because of the crowding of images, and omitting also three for which the periods exceed a day, the median magnitude of the remaining 61, referred to Mount Wilson standards, is 15.26 ± 0.01, the average deviation for a single star being ± 0.075. Considering only the 30 light curves selected by Bailey as well-determined, the mean median magnitude is 15.25, with an average deviation of ± 0.08.

2. In Messier 15 are 51 known variables,[1] but the light curves have not yet been determined. On three plates the extreme magnitude range of the variables has been measured and found to be:

Maximum	14.98	Extremes of range	1.22
Minimum	16.20	Median magnitude	15.59

Two peculiarly bright stars, suspected of variability, are excluded. The range of variation is in very good agreement with the mean ranges for Messier 3 and 5, and, although the results are not final, the median magnitude is probably correct within a tenth of a magnitude.

3. In the southern cluster ω Centauri three subclasses of cluster variables are recognized. Treating each separately we have, for the stars whose classification is certain, the results of table 2, which are

TABLE 2

SUB-CLASS	NUMBER VARIABLES	MEAN PERIOD	MAXIMUM MAGNITUDE	RANGE OF VARIATION	MEDIAN MAGNITUDE	AVERAGE DEVIATION
		days				
a	33	0.586	12.99	1.11	13.55	±0.09
b	15	0.752	13.11	0.87	13.55	±0.10
c	28	0.395	13.33	0.56	13.61	±0.09
All	76				13.57	±0.10

taken from Bailey's tabulation with only slight modifications and additions.[6] Subclass *a*, as the range of variation suggests, is the type that prevails in Messier 3 and 5. Although the stars of the three groups differ from each other in maximum and in range, as well as in period, the median magnitude is the same. The distribution of deviations again follows closely the probability curve. As before the short period variables are restricted to a small interval of brightness which can be represented with high accuracy by a mean median magnitude. The whole interval observed in ω Centauri is six or seven magnitudes, but no fainter variables are found, and the two or three brighter ones are long-period Cepheids. The magnitudes used for this cluster are not referred to the Mount Wilson system, and, therefore, are not strictly comparable with those of the other clusters. They are probably about right, although the uncertainty may be as much as half a magnitude.

4. Ten stars in Messier 2 have been suspected of variation.[1] Eight have been verified on Mount Wilson plates, but as no comparisons with the Pole have been made, only provisional results are available. Measures on three plates give the extreme variation as one and a third magnitudes, agreeing with all that precedes in showing that the short period variables are confined to a definite limit of magnitude.

These results for the variables in five clusters have an obvious application in the determination of relative parallaxes. We need only the hypothesis, apparently reasonable in the light of the foregoing discussion, that the absolute median magnitudes, which appear so constant in each cluster, are actually identical in all systems. The observed differences in the mean values then become sensitive criteria of distance, and the relative parallaxes of these remote systems can be known with an accuracy which will depend only on the precision with which the photographic magnitudes can be determined.

For instance, we find above that the median magnitude for Messier, 3 is 15.49 ± 0.01, and the corresponding value for Messier 5, 15.25 ± 0.01. Possibly there are small systematic errors due to choice of variables, to errors in the maxima, or to remaining errors of zero point, which are not eliminated in taking the difference. But certainly we know the difference in the two median magnitudes within a tenth of a magnitude, and consequently the difference in distance of these remote and nearly equidistant clusters within 40% of its value; while the difference in distance can be known within 7% of its value, if either cluster is compared with ω Centauri. And once we find the absolute median magnitude of such variables—by no means a hopeless task—the actual distances of all clusters with typical variables can be determined within

5%, an accuracy as yet quite unattainable by direct measurement for any stellar object except the nearest stars.

The derivation of a probable value of the absolute luminosity of cluster-type variables will be given in the extended paper. Provisionally we observe that the absolute median magnitude is probably within the limits -0.5 and $+1.5$ (unit of distance, $\pi = 0''.01$), and on that basis the absolute parallax of four of the above clusters is as follows:

Messier 3.....π between $0''.00006$ and $0''.00016$
Messier 5.....π between 0.00007 and 0.00018
Messier 15.....π between 0.00006 and 0.00015
ω Centauri.....π between 0.00015 and 0.00038

Only a small number of the globular clusters are known to contain short-period variables. The relative distances of the others can be estimated, however, on the basis of a relation, found in the clusters that contain variables, of the median magnitude to the average magnitude of the brightest stars. To derive this relation the photographic magnitudes of all the brighter stars in each cluster were determined, excluding stars more distant from the center than 10′ as possibly not members, and also these within 2′ as too liable to uncertainty of measurement. Of those remaining, the first five in order of brightness were discarded as superposed stars, or as non-typical in luminosity. The mean magnitude of the next 25 stars, a homogeneous group of highly luminous objects, was formed. Thus were obtained the results of table 3, which seem as remarkable and as significant for stellar theories as the phenomenon of constant median magnitude.

TABLE 3

CLUSTER	MEDIAN MAGNITUDE	AVERAGE DEVIATION	NUMBER OF VARIABLES	MEAN MAGNITUDE BRIGHTEST 25 STARS	AVERAGE DEVIATION	MEDIAN MINUS BRIGHTEST	WEIGHT
Messier 3	15.50	±0.08	110	14.14	±0.16	1.35	4
Messier 5	15.26	±0.075	61	13.92	±0.15	1.34	2
Messier 15	15.59		49	14.28	±0.18	1.31	2
Messier 2			8		±0.22	1.40	1
Weighted mean difference						1.35	

In view of these results it is reasonable to believe that, if short-period variables did exist in one of the many globular clusters in which they have not been found, their median magnitudes would average about 1.35 fainter than the average magnitude of the 25 selected brightest

stars. In other words, it is proposed to estimate the distances of clusters lacking typical variables on the basis of their bright stars.

Finally it may be noted that our assumption of the identical luminosity of these variables in the various clusters tacitly implies that the corresponding variables in the galactic system are all of the same absolute magnitude. That leads immediately, and without further hypothesis, to the derivation of highly precise relative parallaxes of the 40 scattered stars of this type, and as some of these variables are extremely faint, the results bear directly on the extent of the general stellar system.

Summary.—(1) The median magnitude of the short-period variables apparently has a rigorously constant value in each globular cluster. (2) These stars also possess essentially identical spectra and color variations. (3) Such phenomena, implying remarkably similar physical conditions for all cluster-type variables, must play an important part in theories of Cepheid variation. (4) The present use of the photographic magnitude observations, however, is to derive accurate relative distances of globular clusters on the basis of the differences, from system to system, of the mean median magnitude. (5) The method extends naturally to all isolated cluster-type variables in the galactic system. (6) By means of a relation between the median magnitude and the magnitudes of the brightest stars in a cluster, the relative parallaxes of the many clusters lacking variables may be estimated with considerable assurance when the magnitudes have been measured. (7) The determination of the absolute luminosity of some of these variables will give immediately absolute individual distances of the clusters and of the forty isolated cluster-type variables of known magnitude and period. (8) A preliminary value of this absolute median magnitude indicates that with one or two exceptions no globular cluster is nearer than thirty thousand light-years ($\pi = 0''.00012$).

[1] Bailey, S. I., *Ann. Obs. Harvard Coll., Cambridge*, **38**, 1902, (1–252), page 2.
[2] Shapley, H., *Mt. Wilson Contrib.*, No. 91; *Astroph. J., Chicago*, **40**, 1914, (443–447).
[3] Bailey, S. I., *Ann. Obs. Harvard Coll., Cambridge*, **78**, Part I, 1913, (1–98).
[4] Shapley, H., *Mt. Wilson Contrib.*, No. 115, 1915, (1–92), pages 8 to 22.
[5] Bailey, S. I., *Ann. Obs. Harvard Coll., Cambridge*, **78**, Part II, 1917, (103–192).
[6] *Ibid.*, **38**, 1902, (1–253).

THE PRINCIPAL AXES OF STELLAR MOTION

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Suppose the stars of known motion transferred to a fixed point O, and allowed to move with their present velocities for a definite length of time. They will have expanded into a cluster, the 'velocity-figure.' According to the older idea that the stars were moving 'at random,' this cluster should be spherical. Since J. C. Kapteyn showed[1] the figure to be elongated, several methods of finding the amount and direction of elongation have been proposed and used. Some of these make use only of the directions of observed proper-motions, leaving the amounts of those motions wholly out of account, thus avoiding the difficulties caused by our ignorance of the distances of most stars; and are based on some hypothesis as to the actual shape of the velocity-figure. The most notable of these are A. S. Eddington's[2] and K. Schwarzschild's.[3] The method here used makes no assumption as to the form of the velocity-figure, and is essentially a process of finding its principal momental axes. It thus belongs to the same *genus* as those used by C. V. L. Charlier,[4] K. W. Gyllenberg,[5] A. S. Eddington and W. E. Hartley,[6] and H. C. Plummer.[7]

Let the coordinates of a star in the velocity-figure, referred to suitable axes through O,—say to the First Point of Aries, to (6h, 0°), and to the North Pole, respectively,—be x', y', z'; the direction cosines of an arbitrarily chosen line be l, m, n; and the projection of the star S upon this line be P. Then the component of motion in the direction l, m, n, is $p = OP = lx' + my' + nz'$. Representing the mean of a quantity by enclosing it in square brackets,

$$[pp] = l^2[x'x'] + m^2[y'y'] + n^2[z'z'] + 2lm[x'y'] + 2ln[x'z'] + 2mn[y'z'] \quad (A)$$

The axis of preference is defined as the line for which $[pp]$ is a maximum; for the axis of avoidance $[pp]$ is a minimum.

Newcomb, in his paper 'on the Principal Planes toward which the Stars Tend to Crowd,' applies to another problem an essentially similar process, and gives a detailed solution by Lagrange's method for a maximum or minimum of (A), from which the procedure of this investigation is adapted. Putting $A = [x'x']$, $B = [y'y']$, $C = [z'z']$, $D = [x'y']$, $E = [x'z']$, $F = [y'z']$, the maximum, minimax, and minimum

values of $[pp]$ are the three roots of the cubic

$$\begin{vmatrix} (A-\lambda) & D & E \\ D & (B-\lambda) & F \\ E & F & (C-\lambda) \end{vmatrix} = 0 \qquad \text{(B)}$$

and the corresponding values of l, m, n, are derived by inserting these roots successively in the set of equations

$$\begin{aligned} (A-\lambda)l + \quad Dm + \quad En &= 0 \\ Dl + (B-\lambda)m + \quad Fn &= 0 \\ El + \quad Fm + (C-\lambda)n &= 0 \end{aligned} \qquad \text{(C)}$$

The coordinates x', y', z' are referred to the unknown center of gravity of the velocity-figure. We can observe only the velocities x, y, z, relative to the moving sun. But we readily find that

$$\begin{aligned} [x'x'] &= [xx] - [x]\cdot[x] & [x'y'] &= [xy] - [x]\cdot[y] \\ [y'y'] &= [yy] - [y]\cdot[y] & [x'z'] &= [xz] - [x]\cdot[z] \\ [z'z'] &= [zz] - [z]\cdot[z] & [y'z'] &= [yz] - [y]\cdot[z] \end{aligned} \qquad \text{(D)}$$

The quantities $[x]$, $[y]$, $[z]$, are the negatives of the coordinates of solar motion, according to the method of Bravais.

We are limited to the use of the proper-motions alone, or the radial velocities alone, since to combine them requires a knowledge of individual parallaxes which we do not possess. The rectangular components of the proper-motion part (tangential to the celestial sphere) of the stellar motions, resolved as before, may be expressed in terms of the proper-motions μ, μ', and the spherical coordinates of the stars, while μ, μ' may be expressed in terms of x, y, z. Designating the tangential part by subscript l, we readily obtain

$$x_1 = x(1-\cos^2\alpha\cos^2\delta) - y\sin\alpha\cos\alpha\cos^2\delta - z\cos\alpha\sin\alpha\cos\delta,$$

and similarly for y_1 and z_1. From these x, x_1, etc., are found by multiplication. The means of these quantities are found by integration over the sphere. They are, cleared of fractions,

$$\begin{aligned} 3[x_1] &= 2[x] & 15[x_1y_1] &= 7[xy] \\ 3[y_1] &= 2[y] & 15[x_1z_1] &= 7[xz] \\ 3[z_1] &= 2[z] & 15[y_1z_1] &= 7[yz] \\ 15[x_1x_1] &= 8[xx] + [yy] + [zz] \\ 15[y_1y_1] &= [xx] + 8[yy] + [zz] \\ 15[z_1z_1] &= [xx] + [yy] + 8[zz] \end{aligned} \qquad \text{(E)}$$

The admissibility of these integrations depends upon certain conditions, which may be stated in the form of three hypotheses:

I. The stars are evenly distributed over the sky.

II. The distribution in distance is the same in different parts of the sky.

III. The motions of the stars are distributed in the same way in all accessible parts of space.

These hypotheses are of course not rigorously true. III is open to objection, but if it is not at least approximately true we have no general problem, merely a number of local problems. After the mean square velocities in the three principal directions are known, failure of II, so far as effect of different mean square parallax in different parts of the sky is concerned, may be taken care of and a new approximation made. The theoretical value of $[x_1x_1] + [y_1y_1] + [z_1z_1]$ can be computed for each region of the sky and compared with the observed values. A difference may be considered as due to the stars being nearer or farther than the average and factors applied accordingly. These approximations converge very rapidly; in fact, the error from the source considered is small, even in the first solution. Errors due to the spread or range of the stars in distance are not corrected by this process, but must usually be negligible. Hypothesis I can be made valid by simply weighting the material proportionally to the area of the sky covered, instead of proportionally to the number of stars.

The material used was the 5943 proper-motions less than 80″ per century from Boss's Preliminary General Catalogue. Of these, 5384 were less than 20″ per century (allowing for solar motion[8]) and were called "S"; the remaining 559 stars of larger proper-motion were called "L." Group S was subdivided according to type of spectrum as follows: B, types Oe5 to B 5, 492 stars; A, B 8 to A 4, 1647; F, A 5 through F, 656; G, 446 stars; K, 1227; M, 223; X, including stars of unknown type—mostly faint—and some of types N, O, and P, 693 stars.

The numerical results appear in the accompanying tables. Table 1 gives, in column 1, the designation of the group; in 2 and 3 the Right Ascension and Declination respectively of the direction in which $[pp]$ is a maximum—the vertex of preferential motion; in 4 and 5, the same for direction of least motion. The third direction, in which $[pp]$ is a minimax, is necessarily at right angles to each of these, hence is not given. The last three columns give the value of $[pp]$ for these three directions. Unit of motion, 1″ per century.

Table 2 gives the solar motion derived at the same time. A and D are the Right Ascension and Declination of the Apex, M the amount of solar motion. The last three columns are given as a means of comparing the various groups. If the means without regard to sign of the

components of motion in the three principal directions be $[u]$, $[v]$, $[w]$, we have

$$[u] = \sqrt{(2\lambda_1/\pi)}, [v] = \sqrt{(2\lambda_2/\pi)}, [w] = \sqrt{(2\lambda_3/\pi)}$$

provided the components are distributed according to the error law. They have merely the broad characters of such distribution,—sufficient to justify the process in the same sense that the customary method of finding the 'probable error,' without inquiring too diligently as to whether the errors of observation are actually distributed in the manner assumed in theory, is justified.

TABLE 1

SYSTEMATIC MOTION

GROUP	POLE OF PREFERENCE		POLE OF AVOIDANCE		MEAN SQUARE MOTIONS		
	A_1	D_1	A_2	D_2	λ_1	λ_2	λ_3
B	100°.6	−38°.4	180°.4	+12°.6	4.58	1.94	1.33
A	92.4	+1.8	182.9	+15.2	25.65	9.15	2.57
F	93.1	+2.1	185.2	+45.9	58.05	28.78	22.66
G	93.8	+3.4	185.4	+24.4	32.26	15.81	9.96
K	93.1	+11.3	191.2	+35.3	29.30	18.71	11.46
M	94.0	+25.3	205.1	+37.2	28.14	12.58	3.48
X	83.9	+30.2	192.6	+29.0	21.08	12.72	5.33
S	91.9	+ 4.2	183.8	+24.9	28.91	14.99	9.89
L	92.5	+35.3	211.7	+34.5	745.6	261.4	194.7
All	92.3	+16.9	184.7	+ 8.1	107.3	58.7	31.8

TABLE 2

SOLAR MOTION

GROUP	NUMBER	SOLAR APEX			MEAN MOTIONS		
		A	D	M	[u]/M	[v]/M	[w]/M
B	492	274°.2	+39°.3	2".33	0.73	0.47	0.40
A	1647	266.7	+27.9	4.03	1.00	0.60	0.32
F	656	262.9	+27.0	6.36	0.96	0.67	0.60
G	446	256.6	+45.8	2.60	1.74	1.22	0.97
K	1227	272.5	+38.4	4.02	1.08	0.86	0.67
M	223	272.2	+45.7	3.45	1.23	0.82	0.43
X	693	271.4	+42.1	3.06	1.20	0.93	0.60
S	5384	269.0	+32.1	3.83	1.12	0.81	0.66
L	559	269.3	+31.5	22.17	1.01	0.61	0.52
All	5943	269.1	+32.3	5.42	1.53	1.13	0.83

In regard to these results the following points seem especially worth noting:

1. All groups show markedly unequal motions in three principal directions. In all groups the direction of greatest motion, or axis of preference, lies near the plane of the galaxy not far from its intersection with the equator, the axis of avoidance is nearly perpendicular to the galaxy.

Using radial velocities Gyllenberg[5] finds much the same axis of preference, but the other two axes practically interchanged. So do Eddington and Hartley[6] for later types, but their results for early types agree better with table 1. Are these differences real, in the sense of being typical of what may be expected whenever radial velocities and proper-motions are compared? If so, what peculiarity of distribution or of motions in space, out of the many that might easily be suggested, lies back of it? Here is a subject well worth investigation.

2. The separation of the poles of preference into two groups corresponding respectively to 'early' and 'late' types,[9] is confirmed. There seems to be some evidence of progression of the same character within each of the two groups of types, as if with advance of type the 6-hour vertex moved northward. With advance of type there is an increase of stellar as compared with solar motion; that in the preferred direction, however, increasing less than the other two components.

3. The apices of solar motion show the division into two groups already found by several investigators; A and F having apices in smaller R. A. and Decl. than K, M, and X. B seems to resemble the latter group. G is anomalous.

These results agree rather closely with those found by L. Boss by Airy's method,[10] and by the writer by Schwarzschild's method,[9] all three being based upon practically the same material. This agreement will be found to extend even to details.

4. The distribution of stars of type B is so peculiar that to weight the material according to area covered would have been unfair. The integrations which give equations (E) can be made over zones parallel to the galaxy, and the resulting relations used to correct each zone for perspective separately, the material then being combined with weights proportional to the number of stars. The results in the tables were thus found; a check solution by the ordinary method gave for the vertex (122°,–36°).

The vertex of preferential motion is near the antapex of solar motion. This may in part be a manifestation of the tendency, noted above, for the earlier types to have more southerly vertices. In part it is due to

the fact that the stars' 'spread' in distance tends to produce a false elongation of the velocity-figure in the direction apex-antapex. This tendency is stronger the greater the spread, the smaller the velocity-figure relative to solar motion, and the more nearly spherical the figure. The last two causes, and possibly the first, operate strongly in the B stars. There is no appearance of such an effect in the other groups.

5. The separation of groups "S" and "L" was made some years ago, for a different purpose, and according to a method which made allowance for solar motion relative to group S, but not enough for L, and made none for preferential motion. Consequently, while S no doubt represents fairly well the smaller motions, L does not truly represent the large ones. In group "All," while the large motion characters are diluted by large numbers of small motion stars, the former have individually much more weight than the latter; this therefore is probably more representative that L of the large proper-motion stars. A redivision into small and large proper-motions, on principles better adapted to the purpose of this investigation, is in progress.

[1] *London, Rep. Brit Ass.*, 1905, (257).
[2] *London Mon. Not. R. Astr. Soc.*, **67**, 1906, (34).
[3] *Göttingen, Nachr., Math.-Phys. Kl.*, 1907, (614); 1908 (191).
[4] *Medd. Lunds Astr. Obs.*, (Ser. 2), No. 9, 1913.
[5] *Ibid.*, No. 59, and (Ser. 2), No. 13, 1914–15.
[6] *London Mon. Not. R. Astr. Soc.*, **75**, 1915, (521).
[7] *Ibid.*, **76**, 1915, (121).
[8] *Astr. J. Boston*, **612**, 1910, (98).
[9] *Ibid.*, **676**, 1915.
[10] *Ibid.*, **614**, **623–4**, 1910–11.